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Lower bounds for eigenvalues of the Dirac operator on surfaces of rotation

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Abstract

In this paper we will prove new lower bounds for the first eigenvalue of the Dirac operator on two-dimensional Riemannian manifolds diffeomorphic to S^2 with an isometric S^1-action. We show examples, where this new bound improves the known lower bounds and coincides in the limit with the known upper bounds. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

On closed Riemannian spin manifolds the spectrum of the Dirac operator is discrete and real. There are a few examples where the spectrum is well known, e.g. the flat tori [12], spheres of constant curvature [17] and spherical space forms [4] (for an overview and more examples cf. [2]). For the sphere S^n of constant sectional curvature 1 the Dirac operator has the eigenvalues

$$\pm\left(\frac{n}{2}+k\right), \quad k \geq 0,$$

with multiplicity $2^{[n/2]}\binom{k+n-1}{k}$.

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In general there are only estimates for the eigenvalues of the Dirac operator. For an n-dimensional closed Riemannian spin manifold M^n with scalar curvature R, Friedrich has proved in [13] that the eigenvalues satisfy

$$\lambda^2 \geq \frac{1}{4}\frac{n}{n-1} R_{\min}, \tag{1}$$

with $R_{\min} := \min\{R(m) | m \in M^n\}$.

For two-dimensional manifolds (M^2, g) of genus zero the Dirac operator has no kernel, because any metric g on S^2 is conformally equivalent to the standard metric, and the inequality

$$\lambda^2 \geq \frac{4\pi}{\mathrm{vol}(S^2, g)} \tag{2}$$

holds for the eigenvalues (cf. [5,9,16]).

There are also some upper bounds for the first positive eigenvalue of the Dirac operator, e.g. in [8] it is shown that for compact connected Riemannian spin manifolds of dimension $2m$ and positive sectional curvature K with maximum $K_{\max}$ the inequality

$$\lambda^2 \leq 4^{m-1} m K_{\max}$$

for the first positive eigenvalue holds.

Another intrinsic upper bound for Riemannian manifolds M^2 diffeomorphic to S^2 is given in [1]: Let

$$\delta_c^{\mathrm{Dirac}}(M^2, g) = \inf \left\{ \int_{S^2} \frac{|\mathrm{grad}(h_\Phi)|^2}{h_\Phi^2} \, \mathrm{d}S^2 : \Phi \in U(S^2, M^2) \right\},$$

where $U(S^2, M^2)$ denotes the set of all uniformisation maps preserving the orientation. Then

$$\lambda_1^2 \leq \frac{4\pi}{\mathrm{vol}(M^2, g)} + \frac{\delta_c^{\mathrm{Dirac}}(M^2, g)}{\mathrm{vol}(M^2, g)}. \tag{3}$$

For manifolds M^n, which are isometrically immersed in $\mathbb{R}^{n+1}$, extrinsic upper bounds are known (e.g. [8,10]). In [6,14] the following inequality for the first positive eigenvalue of the Dirac operator on $M^n \hookrightarrow \mathbb{R}^{n+1}$ with the induced spin structure on M^n is shown

$$\lambda^2 \leq \frac{n^2 \int H^2 \, \mathrm{d}M}{4 \, \mathrm{vol}(M)}, \tag{4}$$

where H denotes the mean curvature.

In [1] the inequality

$$\lambda^2 \leq \frac{\int_{M^2} H^2 (f^2 + G^2(f)) \, \mathrm{d}M^2 + \int_{M^2} |\mathrm{grad}\, f|^2 (1 + [G'(f)]^2) \, \mathrm{d}M^2}{\int_{M^2} (f^2 + G^2(f)) \, \mathrm{d}M^2}, \tag{5}$$

where $f : M^2 \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ are smooth functions and G' denotes the derivative of G, has been proved.

In this paper we give new lower bounds for the eigenvalues of the Dirac operator on two-dimensional manifolds of genus 0 with isometric S^1-action. We will show the following:

Theorem 1. *Let (M^2, g) be a closed Riemannian manifold of genus 0 with an isometric S^1-action with maximum length of orbits $2\pi f_{\max}$. Then*
1. $\lambda \geq 1/2f_{\max}$.
2. The multiplicity of an eigenvalue λ_n, $n > 0$, with

$$\frac{2n-1}{2f_{\max}} \leq \lambda_n \leq \frac{2n+1}{2f_{\max}}$$

is at most $2n$.

On the one hand there are many examples, where this estimate improves the known lower bounds for the Dirac operator on surfaces, and on the other hand, it shows that the known upper bounds and the new lower bound coincide in the limit for the ellipsoid. More exactly:

For the ellipsoid $E(a)$ with axes $(a, 1, 1)$ with $a > 1$, the minimal scalar curvature is $1/a^2$ and the volume is $2\pi + (4\pi a^2/\sqrt{a^2-1}) \arcsin \sqrt{(a^2-1)/a^2}$. Therefore the estimates (1) and (2) for the smallest positive eigenvalue $\lambda(a)$ on the ellipsoid $E(a)$ give no lower bound in the limit $a \to \infty$, whereas Theorem 1 shows

$$\lambda(a) \geq \tfrac{1}{2} \quad \text{for every } a > 0.$$

On the other hand, the inequalities (4) and (5) give

$$\overline{\lim_{a\to\infty}} \lambda(a) \leq \tfrac{1}{2}$$

(cf. [1]), which is the optimum result.

In Section 2 we will give some general facts about the Dirac operator on surfaces of rotation and calculate the eigenvalue equation in special coordinates, in Section 3 we will use this equation to get the estimate.

2. The Dirac operator on surfaces of rotation

Let (M, g) be a two-dimensional oriented Riemannian manifold $M \cong S^2$ with an isometric S^1-action, whose principal orbits are free. Then $M/S^1 \cong [0, 1]$, the dense submanifold M_0 of principal orbits is an S^1-fibre bundle over $M_0/S^1 \cong (0, 1)$ (cf. [3, Proposition 8.2]) and the two orbits, which are not free are fixed points.

Let $\gamma : I = (0, L) \to M_0$ be a section of $M_0 \to M_0/S^1$ orthogonal to the fibres parametrised by arc length and $f : I \to \mathbb{R}^+$, $f(t) = (\mathrm{vol}(S^1 \cdot \gamma(t)))/2\pi$, then

$$(M_0, g) = (I \times S^1,\ \mathrm{d}t^2 + f^2(t)\,\mathrm{d}\phi^2) =: I \times_f S^1.$$

For given f the closure of $I \times_f S^1$ is a C^∞-manifold iff $\dot{f}(0) = -\dot{f}(L) = 1$ and f extends at zero to an odd smooth function and similarly at L.

Example. For the ellipsoid with axes (a, b, b) we have $f(t) = b \sin x(t)$, with $t(x) = \int_0^x \sqrt{a^2 \sin^2\phi + b^2 \cos^2\phi}\,\mathrm{d}\phi$, $t \in (0, L)$, $L := \int_0^\pi \sqrt{a^2 \sin^2\phi + b^2 \cos^2\phi}\,\mathrm{d}\phi$.

The frame-bundle P of $I \times_f S^1$ is trivialised by the canonical section $s := (\partial_t, \partial_\phi/|\partial_\phi|)$. S^2 has exactly one spin structure given by the Hopf bundle. Therefore the spin bundle of M is given by

$$Q|I \times_f S^1 \cong I \times S^1 \times S^1 \to P, \quad (t, z_1, z_2) \mapsto (t, z_1, z_1 z_2^2)$$

and the identifications $(0, \phi, z) \sim (0, 0, z)$ and $(1, \phi, z) \sim (1, 0, \mathrm{e}^{-\mathrm{i}\phi} z)$ at the poles.

The section s does not lift to a global section of $Q|I \times_f S^1$. A local lift $\tilde{s}$ of s satisfies $\tilde{s}(t, 0) = -\tilde{s}(t, 2\pi)$ and the section $\tilde{s}\mathrm{e}^{\mathrm{i}\phi/2}$ is globally defined on $I \times_f S^1$. In a neighbourhood of the poles $\tilde{s}\mathrm{e}^{\mathrm{i}\phi/2}$ and $\tilde{s}\mathrm{e}^{-\mathrm{i}\phi/2}$, respectively, are smooth local sections of the spin-bundle over M.

The Levi-Civita connection ∇ on $I \times_f S^1$ is given by

$$\left\langle \nabla \partial_t, \frac{\partial_\phi}{|\partial_\phi|} \right\rangle = \frac{\dot{f}(t)}{f(t)} \sigma^2,$$

with $\sigma^2 = \langle \partial_\phi/|\partial_\phi|, \cdot \rangle$.

The spinor derivative for a local spinor

$$[\tilde{s}, \psi], \quad \text{with } \psi =: \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in C^\infty(I \times_f S^1 \backslash (t, 1), \mathbb{C}^2)$$

is given by

$$\nabla\psi = \mathrm{d}\psi + \frac{\mathrm{i}}{2} \frac{\dot{f}}{f} \begin{pmatrix} -\psi_1 \\ \psi_2 \end{pmatrix} \sigma^2$$

and therefore the Dirac operator by

$$D\psi = \begin{pmatrix} (1/f)\psi_2' + \mathrm{i}\dot{\psi}_2 + (\mathrm{i}/2)(\dot{f}/f)\psi_2 \\ -(1/f)\psi_1' + \mathrm{i}\dot{\psi}_1 + (\mathrm{i}/2)(\dot{f}/f)\psi_1 \end{pmatrix},$$

where $\psi_i' = \partial_\phi \psi_i$ and $\dot{\psi}_i = \partial_t \psi_i$ (for details concerning the Dirac operator see [11] or [15]).

The Dirac operator preserves the Fourier decomposition of ψ, so we restrict to spinors of the form $\psi(t, \phi) = \theta(t)\mathrm{e}^{\mathrm{i}k\phi}$ with $\theta =: \binom{\theta_1}{\theta_2} \in C^\infty(I, \mathbb{C}^2)$.

Then $\psi = \theta \mathrm{e}^{\mathrm{i}k\phi}$ defines a spinor on $I \times_f S^1$ iff $k = (2n+1)/2, n \in \mathbb{Z}$ and on these spinors the Dirac operator acts as

$$D\psi = \mathrm{i} \begin{pmatrix} \dot{\theta}_2 + (1/2f)(2k + \dot{f})\theta_2 \\ \dot{\theta}_1 + (1/2f)(-2k + \dot{f})\theta_1 \end{pmatrix} \mathrm{e}^{\mathrm{i}k\phi}.$$

A spinor $\psi = \theta(t)e^{ik\phi} \in C^\infty(I \times_f S^1 \backslash (t, 1), \mathbb{C}^2)$ is globally defined on M iff $\theta_1(t) e^{i\phi(k-1/2)}$, $\theta_1(t-1)e^{i\phi(k+1/2)}$, $\theta_2(t)e^{i\phi(k+1/2)}$, and $\theta_2(t-1)e^{i\phi(k-1/2)}$ define smooth functions at 0, since $\tilde{s}e^{i\phi/2}$ is a smooth local section at $0 \times S^1$ and $\tilde{s}e^{-i\phi/2}$ at $1 \times S^1$. This means that θ_1 vanishes at zero of order $|k - \frac{1}{2}|$ and at 1 of order $|k + \frac{1}{2}|$, θ_2 vanishes at zero of order $|k + \frac{1}{2}|$ and at 1 of order $|k - \frac{1}{2}|$.

If (M, g) is a Riemannian manifold with isometric S^1-action, then there exists in every neighbourhood of g in the C^1-topology a metric $\tilde{g}$ such that $(M, \tilde{g})$ is the closure of $I \times_{\tilde{f}} S^1$ with an analytic function $\tilde{f}$.

In [7] it is shown that a small deformation of the metric in the C^1-topology does not change the eigenvalues of the Dirac operator too much, more exactly.

Lemma 1 [7, Proposition 7.1]. *For $\epsilon > 0$, $\Lambda > 0$ there exists a C^1-neighbourhood of g, such that for any $\tilde{g}$ in this neighbourhood with associated Dirac operator $\tilde{D}$ and any $\lambda \in [-\Lambda, \Lambda]$ we have*

$$\dim E_{\{\lambda\}}(D) \leq \dim E_{[\lambda-\epsilon,\lambda+\epsilon]}(\tilde{D}) \leq \dim E_{[\lambda-2\epsilon,\lambda+2\epsilon]}(D),$$

where $E_{[a,b]}$ is the direct sum of eigenspaces of D for the eigenvalues $\lambda \in [a, b]$.

So we will restrict in the following to $I \times_f S^1$ with analytic f.

3. Lower bounds for eigenvalues

We consider the Dirac operator on an S^1-manifold which is the closure of $I \times_f S^1$ with analytic $f : I \to \mathbb{R}$. As shown in Section 2 the eigenvalue equation for a local spinor $\psi = \theta e^{i\phi k}$ is given by

$$\lambda\theta_1 = i\left(\dot{\theta}_2 + \frac{1}{2f}(2k + \dot{f})\theta_2\right), \tag{6}$$

$$\lambda\theta_2 = i\left(\dot{\theta}_1 + \frac{1}{2f}(-2k + \dot{f})\theta_1\right), \tag{7}$$

with boundary conditions $|\theta_i(0)| < \infty$, $|\theta_i(L)| < \infty$. These boundary conditions already ensure that θ describes an L^2-spinor and thus as a solution of the eigenvalue equation a C^∞-spinor.

We restrict to the case $\lambda > 0$ and $k > 0$: If $\binom{\theta_1(t)}{\theta_2(t)}e^{i\phi k}$ is an eigenspinor with eigenvalue λ, then also $\binom{\theta_2(t)}{\theta_1(t)}e^{-i\phi k}$, and $\binom{\theta_1(t)}{-\theta_2(t)}e^{i\phi k}$ is an eigenspinor with eigenvalue $-\lambda$.

Furthermore we can assume θ_1 as imaginary and θ_2 real.

Lemma 2. *If θ_1 and θ_2 are solutions of* (6) *and* (7), $k > 0$ *and θ_1 and θ_2 are bounded at* 0 *and L, then $|\theta_1(0)/\theta_2(0)| = \infty$ and $|\theta_1(L)/\theta_2(L)| = 0$.*

Proof. If θ_1, θ_2 are solutions of (6) and (7), then

$$\ddot{\theta}_1 = -\frac{\dot{f}}{f}\dot{\theta}_1 + \left(\frac{(2k - \dot{f})^2}{4f^2} - \frac{\ddot{f}}{2f} - \lambda^2\right)\theta_1 \tag{8}$$

and

$$\ddot{\theta}_2 = -\frac{\dot{f}}{f}\dot{\theta}_2 + \left(\frac{(2k+\dot{f})^2}{4f^2} - \frac{\ddot{f}}{2f} - \lambda^2\right)\theta_2. \tag{9}$$

Thus the characteristic equations of Eqs. (8) and (9) at zero are

$$\nu(\nu-1) + \nu - \tfrac{1}{4}(1-2k)^2 = 0,$$
$$\nu(\nu-1) + \nu - \tfrac{1}{4}(1+2k)^2 = 0.$$

Therefore locally at 0 any solution of (8) and (9), which is bounded, is given by

$$\theta_1(z) = C_1 z^{k-1/2}(1+h_1(z)), \qquad \theta_2(z) = C_2 z^{k+1/2}(1+h_2(z)),$$

with analytic h_i and $h_i(0) = 0$. Indeed taking the expansion in power series in (6) we see that if $\theta_1(t)$ vanishes of order m at 0, then $\theta_2(t)$ vanishes of order $m+1$. Thus $|\theta_1(0)/\theta_2(0)| = \infty$. The same argument at L gives $|\theta_1(L)/\theta_2(L)| = 0$. □

Example. For the sphere Eqs. (6) and (7) give

$$\lambda\theta_1 = \mathrm{i}\left(\dot{\theta}_2 + \frac{1}{2\sin t}(2k+\cos t)\theta_2\right), \tag{10}$$

$$\lambda\theta_2 = \mathrm{i}\left(\dot{\theta}_1 + \frac{1}{2\sin t}(-2k+\cos t)\theta_1\right), \tag{11}$$

with the eigenfunctions $\theta_1(t) = \mathrm{i}\cos(t/2)$ and $\theta_2(t) = \sin(t/2)$ for $\lambda = 1$ and $k = \frac{1}{2}$.

If θ_1 and θ_2 are solutions of (6) and (7) which are bounded at 0 and L, then $\mathrm{i}(\theta_1/\theta_2)$ is a (possibly singular) solution of the real Riccati equation

$$\dot{z} = \lambda z^2 + \frac{2k}{f}z + \lambda, \tag{12}$$

with boundary conditions

$$|z(0)| = \infty, \qquad z(L) = 0. \tag{13}$$

Lemma 3. *If* (12) *has a solution with boundary conditions* (13), *then* $\lambda \geq k/f_{\max}$ *with* $f_{\max} := \max_{t\in[0,L]} f(t)$.

Proof. Consider the t-dependent vectorfield v_t on $\mathbb{R}$ given by

$$v_t(z) := \lambda z^2 + \frac{2k}{f(t)}z + \lambda \quad \text{for } t \in (0, L).$$

$\gamma : (0, L) \to \mathbb{R}$ is a solution of (12) iff $\dot{\gamma}(t) = v_t(\gamma(t))$.

For $z \geq 0, k > 0, \lambda > 0$ the vectorfield v_t satisfies $v_t(z) > 0$ for every $t \in (0, L)$. This means that for every solution γ of (12) and $t_0 \in (0, L)$ with $\gamma(t_0) \geq 0$ the inequality $\dot{\gamma}(t_0) > 0$ holds.

If $\lambda < k/f_{\max}$ than $v_t(-1) = 2(\lambda - k/f(t)) < 0$ for every $t \in (0, L)$, which means that every solution γ of (12) and $t_0 \in (0, L)$ with $\gamma(t_0) = -1$ satisfy $\dot{\gamma}(t_0) < 0$.

Therefore for every solution $\gamma : (0, L) \to \mathbb{R}$ of (12) with $\gamma(L) = 0$ we get $-1 < \gamma(0) < 0$ if $0 < \lambda < k/f_{\max}$. □

For the Dirac operator this yields the following result:

Theorem 2. *Let $M \cong S^2$ be a surface of rotation with $M_0 = I \times_f S^1$, then*

1. *The smallest positive eigenvalue $\lambda_{|k|+1/2}$ with an eigenspinor locally given by $\psi = \theta e^{i\phi k}$ satisfies $\lambda_{|k|+1/2} \geq |k|/f_{\max}$, especially the smallest eigenvalue satisfies*

$$\lambda_1 \geq \frac{1}{2f_{\max}}.$$

2. *The multiplicity of an eigenvalue $\lambda_{|k|+1/2}$ with $|k|f_{\max} \leq \lambda_{|k|+1/2} \leq (|k|+1)/f_{\max}$ is at most $2|k|+1$.*

Especially for the ellipsoid with axes (a, b, b) we get

$$\lambda_1 \geq \frac{1}{2b} \quad \text{for every } a > 0.$$

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References

[1] I. Agricola, Th. Friedrich, Upper bounds for the Dirac operator on surfaces, J. Geom. Phys. 30 (1999) 1–22.
[2] B. Amann, C. Bär, The Dirac operator on nilmanifolds and collapsing circle bundles, Ann. Glob. Anal. Geom. 16 (1998) 221–253.
[3] G.E. Bredon, Introduction to Compact Transformation Groups, Academic Press, New York, 1972.
[4] C. Bär, The Dirac operator on space forms of positive curvature, J. Math. Soc. Jpn. 48 (1994) 69–83.
[5] C. Bär, Lower eigenvalue estimates for Dirac operators, Math. Ann. 193 (1992) 39–46.
[6] C. Bär, Extrinsic bounds for eigenvalues of the Dirac operator, Ann. Glob. Anal. Geom. 16 (1998).
[7] C. Bär, Metrics with harmonic spinors GAFA, vol. 6, no. 6, 1996, pp. 899–942.
[8] H. Baum, An upper bound for the first eigenvalue of the Dirac operator on compact spin manifolds, Math. Zeitschrift 206 (1991) 409–422.
[9] M. Bordoni, Spectral estimates for Schrödinger and Dirac type operators on Riemannian manifolds, Math. Ann. 298 (1994) 693–718.
[10] U. Bunke, Upper bounds of small eigenvalues of the Dirac operator and Isometric immersions, Ann. Glob. Anal. Geom. 9 (1991) 211–243.
[11] Th. Friedrich, Dirac Operatoren in der Riemannschen Geometrie, Vieweg, Braunschweig, 1997.
[12] Th. Friedrich, Die Abhängigkeit des Dirac-Operators von der Spin-Struktur, Colloquium Mathematicum 48 (1) (1984) 57–62.
[13] Th. Friedrich, Der erste eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, Mathematische Nachrichten 97 (1980) 117–146.

[14] Th. Friedrich, On the spinor representation of surfaces in Euclidean 3-space, J. Geom. Phys. 28 (1998) 143–157.
[15] H.-B. Lawson, M.-L. Michelson, Spin Geometry, Princeton University Press, Princeton, NJ, 1989.
[16] J. Lott, Eigenvalue bounds for the Dirac operator, Pac. J. Math. 125 (1986).
[17] S. Sulanke, Berechnung des Spektrums des Quadrates des Dirac-Operators auf der Sphäre und Untersuchung zum ersten Eigenwert von D auf 5-dimensionalen Räumen konstanter positiver Schnittkrümmung, Dissertation, Humboldt-Universität, Berlin, 1981.